２粒子モデルを使ったビーム不安定性に対する空間電荷力の効果の研究

A TWO PARTICLE MODEL FOR STUDY OF EFFECTS OF SPACE-CHARGE FORCE ON BEAM INSTABILITIES

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Abstract

In this report, we present a new two particle model for study of beam instabilities in the presence of the space-charge force. It is a simple expansion of the well-known two particle model for a strong head-tail instability and is still analytically solvable. It leads to a formula for the growth rate as a function of the two dimensionless parameters: the space charge tune shift (normalized by the synchrotron tune) and the wake field strength, Υ. The 3-dimensional contour plot of the growth rate as a function of those two dimensionless parameters reveals stopband structures. Many simulation results generally indicate that beam instability can be damped by a weak space-charge force, but the beam becomes unstable again when the space charge force is further increased. The new two particle model indicates a similar behavior. The damping of beam instabilities in weak space-charge regions and its loss in strong space-charge regions are explained by a combination of linearized coherent kicks between the space-charge force and the wake fields, not by an incoherent tune spread due to the non-linearity of the space-charge force.

1. INTRODUCTION

In low energy high-intensity hadron machines, the space-charge tune shift is an important parameter in the design and operation of the machines. The space-charge force is also believed to affect the behavior of beam instabilities. Many theoretical and simulation studies have been made for a better understanding of their interplay [1-7]. They generally indicate that beam instability can be damped when the space-charge force is weak, but the beam becomes unstable again when it becomes too strong. The mechanism of this loss of the damping effect due to a strong space-charge force has not been well understood so far. If the damping of beam instabilities is caused by the betatron tune spread (Landau damping) due to the non-linearity of the space-charge force, one may naively think that a stronger space-charge force will be more effective in damping of beam instabilities. However, many simulation results show the contrary. This inversion may suggest that the damping phenomenon of beam instabilities in a weak space-charge region may come from a different mechanism.

The two particle model has been applied to illustrate the mechanism of the strong head-tail instability (or the transverse mode-coupling instability [8]) in a very simple but insightful way [Reference [9]: Chao, page 179]. This two particle model can provide a superb framework for study of the space-charge force on beam instabilities just by adding new space-charge terms on top of the existing wake potential ones. The crucial points in this new model are that the resulting equations of motion need to be analytically solvable and the final form of solutions should be a continuous expansion from the one dimensional (the wake strength only) case to two dimensional (the wake strength and the space-charge strength) case.

We briefly summarize Chao's no space-charge model in Section II to review the premise and the solution techniques of the original two particle model, and derive some useful formulae for later use. In Section III and IV, we show solutions and stability diagrams for the weak and strong space-charge cases, respectively. The procedure to identify unstable regions and to compute the growth rate is summarized in Section V. Contour plots of the growth rate are presented in flat and 3-dimensional ways. The paper is concluded with its findings in Section VI.

2. NO SPACE-CHARGE CASE

Let us first review the premise and treatment of Chao's original two particle model by closely following his text book. We assume that a beam is made of two macro-particles, each with charge of $Ne/2$ and each executing synchrotron and betatron oscillations. We assume that their synchrotron oscillations have equal amplitude, but opposite phases. As for the betatron oscillations, we make no such assumption. In what follows, we use s, the distance along the circumference, as an independent variable of motion. During the first half of the synchrotron oscillation period, $T_s = 2\pi/\omega_s$, the particle 2 leads the particle 1 on the synchrotron phase space, and only the trailing particle (the particle 1 in the present

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case) receives transverse kicks from wake fields created by the leading particle (the particle 2 in the present case), that is a function of the transverse displacement of the leading particle. For simplicity, we assume that the wake potential is a constant, W_0, independent of the distance between the two particles. The property of the wake potential requires that $W_0 \geq 0$. The equations of motion for the two particles are

$$y_1'' + \left(\frac{\omega_\beta}{c}\right)^2 y_1 = \frac{N r_0 W_0}{2\gamma C} y_2, \tag{1}$$

$$y_2'' + \left(\frac{\omega_\beta}{c}\right)^2 y_2 = 0, \tag{2}$$

where $y' = dy/ds$, ω_β is the betatron angular frequency, c is the speed of light, γ is the Lorentz factor, C is the circumference of the machine, and r_0 is the classical radius of the particle. Similarly during the second half period of the synchrotron oscillation, we have the same equations with indices 1 and 2 exchanged.

The solution for y_2 is simply a free betatron oscillation:

$$\tilde{y}_2(s) = \tilde{y}_2(0) e^{-i\omega_\beta s/c}, \tag{3}$$

where

$$\tilde{y}_2 = y_2 + i\frac{c}{\omega_\beta} y_2'. \tag{4}$$

The solution for y_1 is simplified when the betatron frequency is much larger than the synchrotron one, $\omega_\beta \gg \omega_s$, which is mostly the case. It is then approximately given by

$$\tilde{y}_1(s) = \tilde{y}_1(0) e^{-i\omega_\beta s/c} + i\Upsilon \tilde{y}_2(0) e^{-i\omega_\beta s/c}, \tag{5}$$

where we have defined a positive dimensionless parameter for the wake potential strength

$$\Upsilon = \frac{\pi N r_0 W_0 c^2}{4\gamma C \omega_\beta \omega_s}\ . \tag{6}$$

We can write the solutions for the equations of motion during the period $0 < \frac{s}{c} < T_s/2$ in a matrix form as

$$\begin{bmatrix}\tilde{y}_1\\ \tilde{y}_2\end{bmatrix}_{s=cT_s/2} = e^{-i\omega_\beta T_s/2}\begin{bmatrix}1 & i\Upsilon\\ 0 & 1\end{bmatrix}\begin{bmatrix}\tilde{y}_1\\ \tilde{y}_2\end{bmatrix}_{s=0}. \tag{7}$$

The transfer matrix during the second half of the synchrotron oscillation period, $T_s/2 < \frac{s}{c} < T_s$, is obtained by exchanging the indices 1 and 2 in the above treatment. The total transfer matrix for one full synchrotron oscillation period is then given by

$$\begin{bmatrix}\tilde{y}_1\\ \tilde{y}_2\end{bmatrix}_{s=cT_s} = e^{-i\omega_\beta T_s}\begin{bmatrix}1 & 0\\ i\Upsilon & 1\end{bmatrix}\begin{bmatrix}1 & i\Upsilon\\ 0 & 1\end{bmatrix}\begin{bmatrix}\tilde{y}_1\\ \tilde{y}_2\end{bmatrix}_{s=0}$$
$$= e^{-i\omega_\beta T_s}\begin{bmatrix}1 & i\Upsilon\\ i\Upsilon & 1-\Upsilon^2\end{bmatrix}\begin{bmatrix}\tilde{y}_1\\ \tilde{y}_2\end{bmatrix}_{s=0}. \tag{8}$$

Let us find eigenvalues λ of this matrix by equating it as

$$e^{-i\omega_\beta T_s}\begin{bmatrix}1 & i\Upsilon\\ i\Upsilon & 1-\Upsilon^2\end{bmatrix}\begin{bmatrix}\tilde{y}_1\\ \tilde{y}_2\end{bmatrix}_{s=0} = e^{-i\omega_\beta T_s}\cdot\lambda\mathbf{I}\begin{bmatrix}\tilde{y}_1\\ \tilde{y}_2\end{bmatrix}_{s=0}, \tag{9}$$

where **I** is the unit matrix.

The two eigenvalues are found to be

$$\lambda = \begin{cases} 1-\frac{\Upsilon^2}{2} \pm \sqrt{\frac{\Upsilon^2}{2}\cdot\left(\frac{\Upsilon^2}{2}-2\right)} & if\ \Upsilon^2 \geq 4 \\ 1-\frac{\Upsilon^2}{2} \pm i\sqrt{\frac{\Upsilon^2}{2}\cdot\left(2-\frac{\Upsilon^2}{2}\right)} & if\ \Upsilon^2 \leq 4 \end{cases}. \tag{10}$$

Let us check the stability of the system for the two cases. If $\Upsilon^2 \leq 4$, the square of the absolute value of λ becomes

$$|\lambda|^2 = \left(1-\frac{\Upsilon^2}{2}\right)^2 + \frac{\Upsilon^2}{2}\cdot\left(2-\frac{\Upsilon^2}{2}\right) = 1. \tag{11}$$

Namely, the system is stable. If $\Upsilon^2 \geq 4$, one of the solutions,

$$\lambda = 1-\frac{\Upsilon^2}{2} - \sqrt{\frac{\Upsilon^2}{2}\cdot\left(\frac{\Upsilon^2}{2}-2\right)} \leq -1 \tag{12}$$

with its amplitude larger than one, is unstable. At the threshold value of $\Upsilon^2 = 4$, the eigenvalue λ becomes exactly minus one ($\lambda = -1$). It can be expressed in the complex phase space as

$$\lambda = e^{\pm i\pi}. \tag{13}$$

It indicates that during the full synchrotron oscillation period, the coherent betatron frequency shift, $\Delta\omega_\beta$, generates a betatron phase advance by $\pm\pi$. In other words,

$$(\omega_\beta + \Delta\omega_\beta)\cdot T_s = \omega_\beta \cdot T_s \pm \pi. \tag{14}$$

It implies that the strong head-tail instability occurs by the mode coupling between the two solutions when the difference of their phase advances over one synchrotron period becomes exactly 2π. The growth rate g, when $\Upsilon^2 \geq 4$, is obtained by equating

$$|\lambda| = e^{gT_s} = \sqrt{\frac{\Upsilon^2}{2}\cdot\left(\frac{\Upsilon^2}{2}-2\right)} + \frac{\Upsilon^2}{2} - 1. \tag{15}$$

The formula for the growth rate is

$$g = \frac{1}{T_s}\log\left\{\sqrt{\frac{\Upsilon^2}{2}\cdot\left(\frac{\Upsilon^2}{2}-2\right)}+\frac{\Upsilon^2}{2}-1\right\}. \qquad (16)$$

3. WEAK SPACE-CHARGE CASE

Now, let us introduce the space-charge force into the two particle model. Here, we make the following two approximations:

1. The space-charge force is linear in the relative distance between the two particles (the linear model).
2. The two particles interact continuously and coherently with a space charge force in the transverse plane (the continuous interaction model).

At a low energy where the space-charge force is significant, the space-charge fields spread out angularly with a large spread on the order of 1/γ, unlike a high-energy electron beam where the space-charge fields are Lorentz contracted into a thin disk. In this sense, two particles behave like rods, not point charges, each with a half bunch length of the total beam.

Under this linearized continuous interaction model, we have additional space charge terms in the equations of motion for the period $0 < \frac{s}{c} < T_s/2$ as

$$y_1'' + \left(\frac{\omega_\beta}{c}\right)^2 y_1 = K(y_1 - y_2) + W y_2,$$
$$y_2'' + \left(\frac{\omega_\beta}{c}\right)^2 y_2 = K(y_2 - y_1). \qquad (17)$$

Here, we define W as

$$W = \frac{N r_0 W_0}{2\gamma C} \qquad (18)$$

and K denotes the space-charge kick strength:

$$K = \frac{N r_0}{a^2\beta^2\gamma^3 C}\ , \qquad (19)$$

where a is the transverse beam size and β is the Lorenz β. The properties of the wake potential and the space charge force require that the both W and K are always positive

$$\mathrm{W}, \mathrm{K} \geq 0. \qquad (20)$$

The ratio r=K/W can be expressed in a more familiar way, using ϒ (defined by Eq. (6)) and the space-charge tune shift parameter $\Delta\nu_{sc}$(normalized by the synchrotron tune ν_s), as

$$\mathrm{r} = \frac{\pi}{2\Upsilon}\left(\frac{\Delta\nu_{sc}}{\nu_s}\right). \qquad (21)$$

The transfer matrix during the second half of the synchrotron oscillation period, $T_s/2 < \frac{s}{c} < T_s$, is obtained by exchanging the indices 1 and 2 in Eq. (17).

Hereafter, we assume that $W \geq K$, namely we deal with the weak space-charge case. Using the eigenvalue/eigenvector technique, the equations of motion can be readily solved. The eigenvalues are

$$\lambda = \begin{cases} 1-\frac{\Gamma^2}{2} \pm \sqrt{\frac{\Gamma^2}{2}\cdot\left(\frac{\Gamma^2}{2}-2\right)} & if\ \Gamma^2 \geq 4 \\ 1-\frac{\Gamma^2}{2} \pm i\sqrt{\frac{\Gamma^2}{2}\cdot\left(2-\frac{\Gamma^2}{2}\right)} & if\ \Gamma^2 \leq 4 \end{cases}. \qquad (22)$$

Here we define Γ as

$$\frac{\Gamma^2}{2} = 2\cdot\frac{1-y^2}{y^2}\cdot\frac{\tanh^2\left(\frac{\Upsilon}{2}y\right)}{1-\tanh^2\left(\frac{\Upsilon}{2}y\right)}, \qquad (23)$$

where the wake potential strength parameter ϒ is defined by Eq. (6) and y is given by

$$\mathrm{y} = 2\sqrt{r(1-r)}\ . \qquad (24)$$

The threshold value of ϒ as a function of the ratio r up to r=1 for the weak space-charge case (r=K/W≤1) is plotted in Fig. 1. Unstable regions are shown shaded.

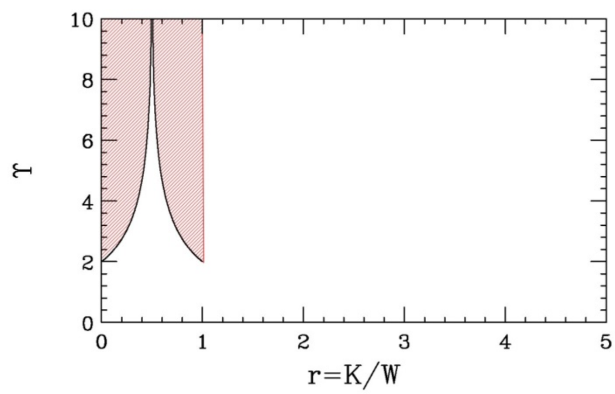

Figure 1: The stability diagram for the weak space-charge case (r=K/W≤1). Unstable regions are shown shaded.

The two eigenvalues for $\Gamma^2 \leq 4$ can be expressed by the left schematic of Fig. 2 in the complex phase plane. The angles between these vectors and the positive real axis are the phase advances of head-tail modes after one synchrotron oscillation period. They line up on the negative real axis (the right schematic of Fig. 2) when Γ is two or when the difference of their phase advances over one synchrotron period becomes exactly 2π.

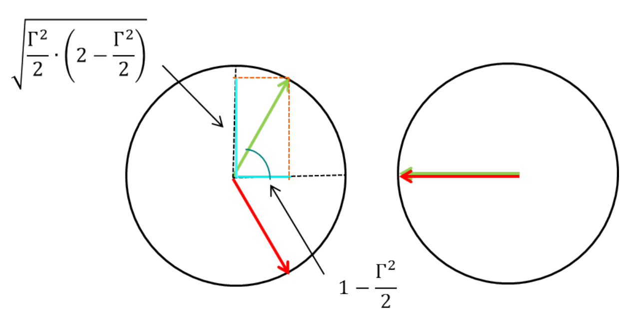

Figure 2: Schematics of the eigenvalues in the complex phase plane. Left for $\Gamma^2 \leq 4$ and right for $\Gamma = 2$.

4. STRONG SPACE-CHARGE CASE

The eigenvalues for the strong space-charge case (r=K/W≥1) can be expressed by Eq. (22) as well, though the parameter Γ is now defined by

$$\frac{\Gamma^2}{2} = 2 \cdot \frac{1+y^2}{y^2} \cdot \frac{\tan^2\left(\frac{\Upsilon}{2}y\right)}{1+\tan^2\left(\frac{\Upsilon}{2}y\right)}. \tag{25}$$

Here the parameter y is defined by

$$y = 2\sqrt{r(r-1)}. \tag{26}$$

The threshold value of Υ as a function of the ratio r for the strong space-charge case (r=K/W≥1) is plotted in Fig. 3, together with the previous weak space-charge case (r≤1). Unstable regions are shown shaded.

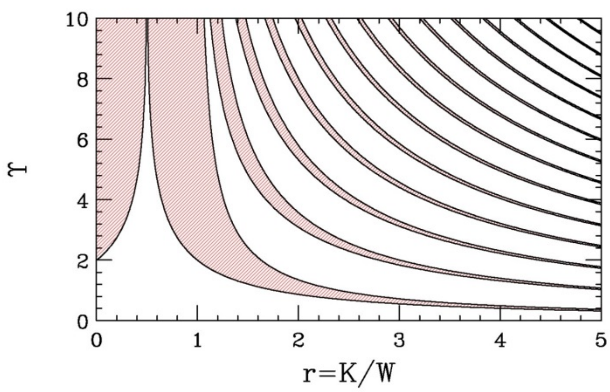

Figure 3: The stability diagram for the strong space-charge case (r=K/W≥1). The stability diagram for the weak space-charge case (r=K/W≤1) is also plotted for completion. Unstable regions are shown shaded.

5. PROCEDURETOIDENTIFYUNSTABLE REGIONS AND GROWTH RATE

We now have the total solution for both the weak and strong space-charge regions. We can calculate the stability diagram using the following steps for given Υ and:

1. Calculate the ratio r

$$r = \frac{K}{W} = \frac{\pi}{2\Upsilon}\left(\frac{\Delta\nu_{sc}}{\nu_s}\right). \tag{27}$$

2. If r≤1(the weak space-charge case), calculate y as

$$y = 2\sqrt{r(1-r)}. \tag{28}$$

2.1 If

$$\tanh^2\left(\frac{\Upsilon}{2}y\right) \le y^2, \tag{29}$$

the beam is stable.

2.2 If

$$\tanh^2\left(\frac{\Upsilon}{2}y\right) \ge y^2, \tag{30}$$

the beam is unstable and the growth rate is given by

$$g = \frac{1}{T_s}\log\left\{\sqrt{\frac{\Gamma^2}{2}\cdot\left(\frac{\Gamma^2}{2}-2\right)} + \frac{\Gamma^2}{2} - 1\right\}, \tag{31}$$

where

$$\frac{\Gamma^2}{2} = 2 \cdot \frac{1-y^2}{y^2} \cdot \frac{\tanh^2\left(\frac{\Upsilon}{2}y\right)}{1-\tanh^2\left(\frac{\Upsilon}{2}y\right)}. \tag{32}$$

3. If r≥1 (the strong space-charge case), calculate y as

$$y = 2\sqrt{r(r-1)}. \tag{33}$$

3.1 If

$$\tan^2\left(\frac{\Upsilon}{2}y\right) \le y^2, \tag{34}$$

the beam is stable.

3.2 If

$$\tan^2\left(\frac{\Upsilon}{2}y\right) \ge y^2, \tag{35}$$

the beam is unstable. The growth rate is given by Eq. (31), though the parameter Γ is now defined by

$$\frac{\Gamma^2}{2} = 2 \cdot \frac{1+y^2}{y^2} \cdot \frac{\tan^2\left(\frac{\Upsilon}{2}y\right)}{1+\tan^2\left(\frac{\Upsilon}{2}y\right)}. \tag{36}$$

We calculate the growth factor $g \times T_s$ as a function of Υ and $\Delta\nu_{sc}/\nu_s$ and plot it in a flat contour plot in Fig. 4 and in a 3-dimension contour plot in Fig. 5. The three parametersg $\times T_s$, Υ and $\Delta\nu_{sc}/\nu_s$ are all dimensionless parameters and these contour plots are universal.

One can see stopband structures in the stability diagrams Figs. 4 and 5. The lowest two stopbands are strongest (namely, very unstable), while other higher-order stopbands are considerably weaker.

The appearance of the stopbands can be explained as follows. A mode-coupling instability takes place when the two eigenvalues line up on the negative real axis in the complex phase plane (see Fig. 2), in other words, when the difference of their phase advances over one synchrotron period is an odd integer times 2π. In the strong space-charge region ($\Delta\nu_{sc} \gg \nu_s$), the tune shift of one solution is close to $-\Delta\nu_{sc}$, while other solution has almost no tune shift. The mode-coupling condition mentioned in the above corresponds to the case when $\Delta\nu_{sc}$ takes values around an odd integer times ν_s. That is why the stopbands always start with $\Delta\nu_{sc}/\nu_s$ equal to odd integers at small Υ in Fig. 4. Pure space-charge oscillations are stable, but even slight inclusion of wake filed effects to them can make such oscillations unstable.

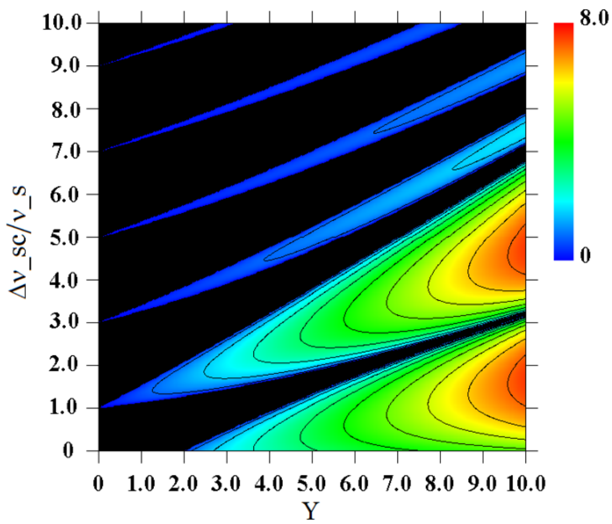

Figure 4: Flat contour plot for the growth factor $g \times T_s$ as a function of Υ and $\frac{\Delta\nu_{sc}}{\nu_s}$.

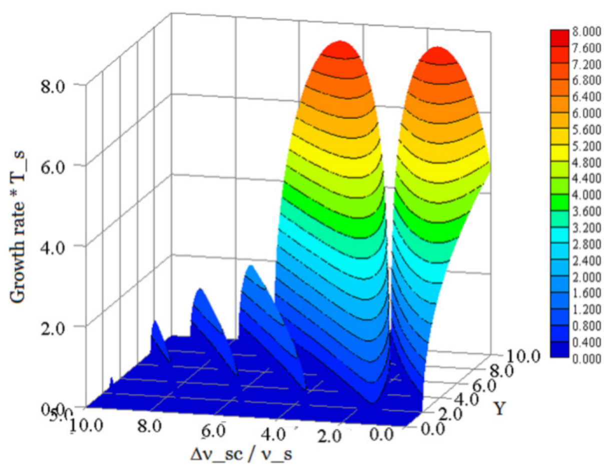

Figure 5: 3-dimensional contour plot for the growth factor $g \times T_s$ as a function of Υ and $\frac{\Delta\nu_{sc}}{\nu_s}$.

6. FINDINGS AND CONCLUSIONS

Let us investigate how the space-charge force affects the strong head-tail instability. Take a case of $\Upsilon = 4$ where the beam is unstable without the space-charge force ($\frac{\Delta\nu_{sc}}{\nu_s} = 0$). Figure 6 shows the growth rate ($\times\, T_s$) as a function of the space-charge tune shift (normalized by the synchrotron tune) at $\Upsilon = 4$. If we gradually increase the space-charge force, the beam moves from the unstable region (the lowest stopband) to the stable region (the lowest passband). However, if we further increase the space-charge force, the beam enters another unstable region (the second lowest stopband). The maximum growth rate in this unstable region is comparable to that for no space-charge case. One may conclude that the space-charge force loses its damping effect when it is too strong. In fact, many theoretical and simulation studies show similar behaviors. If we further increase the space-charge force, the beam would be stable again. However, it is not clear if we can achieve this state in reality or computer simulations since such strong space-charge force may expand the beam size, with a result of reduction of the space-charge tune shift by itself.

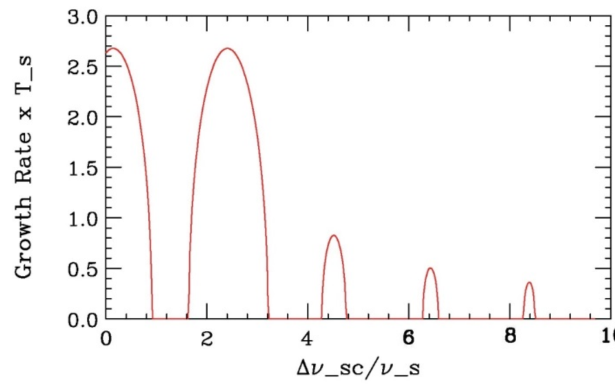

Figure 6: The growth rate ($\times\, T_s$) as a function of the space-charge tune shift (normalized by the synchrotron tune) at $\Upsilon = 4$.

The present two particle model has no tune spread effect, since the space-charge force is linearized in the transverse position. The damping of beam instabilities with a weak space-charge force is caused by pure coherent kicks of the space-charge force in a way to partially neutralize the coherent wake field kicks. The loss of the damping effect with a strong space-charge force is due to an unfavorable combination between the coherent space-charge kicks and the coherent wake field kicks. As stated in the Introduction, if the damping of beam instability with a weak space-charge force is caused by the tune spread (Landau damping) due to the non-linearity of the space-charge force, its loss with a strong space-charge force is hard to explain. The present model suggests that the main damping mechanism of beam instabilities with a weak space-charge force (as observed in many simulations) is linear coherent space-charge kicks, not the Landau damping due to the non-linearity of the space-charge force.

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